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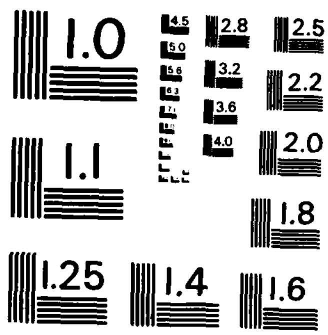

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LIMITING PROPERTIES OF LARGE SYSTEM OF RANDOM LINEAR EQUATIONS*

by

Z. D. Bai

Center for Multivariate Analysis
University of Pittsburgh

Center for Multivariate Analysis

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Z. D. Bai

Center for Multivariate Analysis
University of Pittsburgh

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Fifth Floor, Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

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LIMITING PROPERTIES OF LARGE SYSTEM OF RANDOM LINEAR EQUATIONS

by

Z. D. Bai

The University of Pittsburgh

ABSTRACT

S. Geman and Chi R. Hwang (Z. Wahrscheinlichkeistheoria verw. Gehiete, 1942) proposed a kind of algebraic system of equations and proved the law of large numbers for its solution. In this paper, the conditions to ensure these results are significantly weakened for the law of large numbers. Also, the central limit theorem is shown. For both the law of large numbers and the limit theorem, the only needed assumption is that the random variables have finite second moment.

1. INTRODUCTION

Let $\{\omega_{ij}\}$, $i,j = 1,2,\ldots$, be a collection of independent and identically distributed random variables with zero mean. For each n, $n = 1,2,\ldots$, define W_n to be the $n \times n$ matrix whose (i,j) entry is ω_{ij}. Given a sequence $\alpha_1,\alpha_2,\ldots$, define for each n, $V_n = (\alpha_1,\alpha_2,\ldots,\alpha_n)^T$. Finally, for each n, define a random vector $X_n = (X_{n1},X_{n2},\ldots,X_{nn})^T$ as the solution to the equation

$$X_n = V_n + \frac{1}{n} W_n X_n \tag{1.1}$$

i.e. $X_{n,i} = \alpha_i + \frac{1}{n}\sum_{j=1}^{n} \omega_{ij} X_{nj}$, $j = 1,2,\ldots,n$.

This system of equations plays an improtant rule in large and homogeneous systems of physics (see [1] and [2]). When n is large enough, the solution of (1.1) is usually assumed being "nearly independent" (the so-called chaos hypothesis). This hypothesis was first proved by S. Geman and Chi R. Hwang. They usually assume that $E\omega_{11}^8 < \infty$. For some stronger conclusions, it is even assumed that the characteristic function of ω_{11} has a nondegenerate analytic zone.

In this paper, we relax all these restrictions to the existence of the second moment of ω_{11} and prove somewhat stronger conclusions than that are shown in [1]. Exactly speaking, we get the following theorems.

Theorem 1. Define X_n by (1.1) whenever $I - \frac{1}{n} W_n$ is nonsingular. Otherwise, define $X_n = 0$. Suppose that $E\omega_{11} = 0$ and $E\omega_{11}^2 < \infty$.

1) If $(\alpha_1,\alpha_2,\ldots) \in \ell_\infty$, then

$$\max_{1\le i\le n} |X_{n,i} - \alpha_i| \longrightarrow 0, \text{ a.s. } n \to \infty. \tag{1.2}$$

2) If $\lim_n \alpha_n = 0$, then

$$\sum_{i=1}^{n} (X_{n,i} - \alpha_i)^2 \longrightarrow 0, \text{ a.s. } n \to \infty \tag{1.3}$$

especially for $(\alpha_1,\alpha_2,\ldots) \in \ell_2$,

$$(X_{n,1},\ldots,X_{n,n},0,\ldots) \to (\alpha_1,\alpha_2,\ldots) \quad \text{in } \ell_2, \text{ a.s.} \tag{1.4}$$

If $\alpha_1 = \alpha_2 = \ldots = \alpha$ and if $E\omega_{11} = m$ (instead of zero), Geman and Hwang proved $X_{ni} \to \alpha/1-m$ a.s. under the conditions that $|m| < 1$ and $E|\omega_{11}|^n \le n^{\beta n}$, $\forall\, n \le 2$, for some positive constant β. Corresponding to this, we have

Theorem 2. Assume $\alpha = \alpha_1 = \alpha_2 = \ldots$, $E\omega_{11} = m$. Define X_n by (1.1) whenever $I - \frac{1}{n} W_n$ is nonsingular and define $X_n = 0$ otherwise. If $m \ne 1$ and $E\omega_{11}^2 < \infty$, then $\max\limits_{1\le i\le n} |X_{ni} - \frac{\alpha}{1-m}| \to 0$. a.s.

For the CLT of X_n, we have

Theorem 3. If $E\omega_{11} = 0$, $0 < E\omega_{11}^2 = \sigma^2 < \infty$, $(\alpha_1,\alpha_2,\ldots) \in \ell_\infty$, and $\sum\limits_{i=1}^{\infty} \alpha_i^2 = \infty$. Then for any given integers $(m_1 < m_2 < \ldots < m_k)$,

$$\frac{n}{\sigma\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2}} (X_{n,m_1} - \alpha_{m_1}, X_{n,m_2} - \alpha_{m_2},\ldots, X_{n,m_k} - \alpha_{m_k})^T \longrightarrow N(0,I_k)$$

where I_k denotes the $k \times k$ unit matrix.

2. SOME LEMMAS

We first prove some lemmas.

Lemma 2.1. If $E\omega_{11}^2 < \infty$, then $||(\frac{Wn}{n})^2|| \to 0$ a.s.

where the norm means the Euclidean one, i.e., for $A = (a_{ij})$,

$||A|| = (\sum_i \sum_j a_{ij}^2)^{\frac{1}{2}}$.

Proof. $||(\frac{Wn}{n})^2||^2 = \frac{1}{n^4} \sum_{i,j} (\sum_k \omega_{ik} \omega_{kj})^2$

$$= J_1(n) + J_2(n) + J_3(n) + J_4(n) \tag{2.1}$$

where

$$J_1(n) = \frac{1}{n^4} \sum_{i=1}^{n} \omega_{ii}^4 \tag{2.2}$$

$$J_2(n) = \frac{1}{n^4} \{ \sum_{i\neq j} (\omega_{ij}^2\omega_{ji}^2 + 2\, \omega_{ii}^2\omega_{ij}\omega_{ji} + 2\, \omega_{ii}\omega_{ij}^2\omega_{jj})\} \tag{2.3}$$

$$J_3(n) = \frac{2}{n^4}\{ \sum_{i>j} \sum_k \omega_{ik}^2\omega_{kj}^2 \} \tag{2.4}$$

$$J_4(n) = \frac{2}{n}\{ \sum_{\substack{i\\ i\neq k\\ i\neq \ell}} \sum_{k>\ell} \omega_{ik}\omega_{ki}\omega_{ih}\omega_{hi} + 2 \sum_{\substack{i>j\\ i\neq k}} \sum_{k>h} \omega_{ik}\omega_{kj}\omega_{ih}\omega_{hj} + 2 \sum_{\substack{i>j\\ j\neq h}} \sum_{i>h} \omega_{ii}\omega_{ij}\omega_{ih}\omega_{hj}\} \tag{2.5}$$

It is easy to see that

$E\, J_4(n) = 0$

$E\, J_4\, (n) \leq \frac{4}{n^4} (E\omega_{11}^2)^4$.

By Chebyshev inequality and Borel-Cantelli lemma, we obtain that

$$J_4(n) \to 0 . \quad \text{a.s.} \tag{2.6}$$

By Marcinkiewicz strong law of large numbers, we have

$$J_1(n) = 0(\frac{1}{n^2}) . \quad \text{a.s.} \tag{2.7}$$

and

$$\frac{1}{n^4} \sum_{i\neq j} \omega_{ij}^2\omega_{ji}^2 = 0(\frac{1}{n^2}) . \quad \text{a.s.} \tag{2.8}$$

$$\left|\frac{1}{n^4}\sum_{i\neq j}\omega_{ii}^2\omega_{ij}\omega_{ji}\right| \le \left(\left(\frac{1}{n^4}\right)\sum_{i\neq j}\omega_{ii}^{4}\right)^{\frac12}\left(\left(\frac{1}{n^4}\right)\sum_{i\neq j}\omega_{ij}^2\omega_{ji}^2\right)^{\frac12}$$

$$\le \left(\frac{1}{n^3}\sum_{i=1}^{n}\omega_{ii}^4\right)^{\frac12}\left(\frac{1}{n^4}\sum_{i\neq j}\omega_{ij}^2\omega_{ji}^2\right)^{\frac12}$$

$$= [0(\tfrac{1}{n})]^{\frac12}\,[0(\tfrac{1}{n^2})]^{\frac12} = 0(n^{-3/2}) \quad \text{a.s.} \qquad (2.9)$$

and

$$\left|\frac{1}{n^4}\sum_{i\neq j}\omega_{ii}\omega_{ij}^2\omega_{jj}\right|$$

$$\le \left(\frac{1}{n^4}\sum_{i\neq j}\omega_{ii}^2\omega_{jj}^2\right)^{\frac12}\left(\frac{1}{n^4}\sum_{i\neq j}\omega_{ij}^4\right)^{\frac12}$$

$$\le \left(\frac{1}{n^2}\sum_{i=1}^{n}\omega_{ii}^2\right)\left(\frac{1}{n^4}\sum_{i\neq j}\omega_{ij}^4\right)^{\frac12}$$

$$= 0\left(\frac{1}{n}\right)(0(1))^{\frac12} = 0\left(\frac{1}{n}\right). \quad \text{a.s.} \qquad (2.10)$$

hence

$$J_2(n) = 0\left(\frac{1}{n}\right). \quad \text{a.s.} \qquad (2.11)$$

To prove $J_3(n) \to 0$. a.s. we define

$$\hat{\omega}_{ijn} = \omega_{ij}\, I_{[|\omega_{ij}| < n]}$$

$$\overline{\omega_{ijn}^2} = \hat{\omega}_{ijn}^2 - E\,\hat{\omega}_{ijn}^2$$

and

$$\hat{J}_3(n) = \frac{2}{n^4}\sum_{i>j}\sum_{k}\hat{\omega}_{ikn}^2\,\hat{\omega}_{kjn}^2$$

$$\overline{J_3(n)} = \frac{2}{n^4}\sum_{i>j}\sum_{k}\overline{\omega_{ikn}^2}\;\overline{\omega_{kjn}^2}\,.$$

Notice that $E\,\overline{\omega_{ikn}^2} = 0$, $E\,\overline{\omega_{kjn}^2} = 0$ and that $\overline{\omega_{ikn}^2}$, $\overline{\omega_{kjn}^2}$ are independent of each other for given $i > j$ and k, we know

$$E\,\overline{J_3(n)} = 0$$

and

$$E(\overline{J_3(n)})^2 = \frac{4}{n^8}\sum_{i>j}\sum_{k}(E(\overline{\omega_{11n}^2})^2)^2$$

$$\le 4n^{-5}\,[E\,\hat{\omega}_{11n}^4]^2$$

$$\le 4\,E\,\omega_{11}^2\, n^{-3}\, E\,\hat{\omega}_{11n}^4\,.$$

Therefore, for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} P(|\overline{J_3(n)}| \geq \varepsilon) \leq 4\, E\,\omega_{11}^2\, \varepsilon^{-2} \sum_{n=1}^{\infty} n^{-3}\, E\,\hat{\omega}_{11\,n}^4$$

$$\leq c \sum_{n=1}^{\infty} n^{-3} \Big(\sum_{k=1}^{n} k^2\, E\,\omega_{11}^2\, I_{[k-1 \leq |\omega_{11}^2| < k]} + 1 \Big)$$

$$= c\, [1 + \sum_{k=1}^{\infty} k^2\, E\,\omega_{11}^2\, I_{[k-1 \leq |\omega_{11}^2| < k]} \sum_{n=k}^{\infty} n^{-3}]$$

$$\leq c\, [1 + \sum_{k=1}^{\infty} E\,\omega_{11}^2\, I_{[k-1 \leq |\omega_{11}^2| < k]}]$$

$$\leq c\, [1 + E\,\omega_{11}^2] < \infty. \tag{2.12}$$

here and after, c denotes a positive constant independent of n or k, but it may take different value in each appearance.

From (2.12) we get

$$\overline{J_3(n)} \to 0. \quad \text{a.s.} \tag{2.13}$$

On the other hand, noticing $E\,\hat{\omega}_{11\,n}^2 \leq E\,\omega_{11}^2 < \infty$, we have

$$|\, \overline{J_3(n)} - \hat{J}_3(n)\,| \leq \frac{c}{n^4} \,\Big|\sum_{i=1}^{n-1} \sum_{k=1}^{n} (n-i)\, \overline{\omega_{ikn}^2}\,\Big| + \frac{c}{n^4} \sum_{j=2}^{n} \sum_{k=1}^{n} (j-1)\, \hat{\omega}_{kjn}^2$$

$$\leq \frac{c}{n^3} \sum_i \sum_k \omega_{ik}^2 + \frac{c}{n} \longrightarrow 0, \quad \text{a.s.} \tag{2.14}$$

here the first term tends to zero from Kolmogorov's strong law of large numbers. From (2.13) (2.14) it follows that

$$\lim_{n\to\infty} \hat{J}_3(n) = 0. \quad \text{a.s.} \tag{2.15}$$

Finally, we prove that

$$P(\hat{J}_3(n) \neq J_3(n),\ \text{i.o}) = 0.$$

We have

$$P(\hat{J}_3(n) \neq J_3(n),\ \text{i.o}) \leq \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{2^{m-1} \leq n < 2^m} (\hat{J}_3(n) \neq J_3(n)) \Big)$$

$$\leq \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{2^{m-1} \leq n < 2^m} \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (|\omega_{ij}| > n) \Big)$$

$$\leq \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{2^{m-1} \leq n < 2^m} \bigcup_{i=1}^{2^m} \bigcup_{j=1}^{2^m} (|\omega_{ij}| \geq 2^{m-1}) \Big)$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \sum_{m=k}^{\infty} P\Big(\bigcup_{i=1}^{2^m} \bigcup_{j=1}^{2^m} (|\omega_{ij}| \geq 2^{m-1})\Big) \\
&\leq \lim_{k\to\infty} \sum_{m=k}^{\infty} 2^{2m} P(|\omega_{11}| \geq 2^{m-1}) \\
&= \lim_{k\to\infty} \sum_{m=k}^{\infty} 2^{2m} \sum_{\ell=m}^{\infty} P(2^{\ell-1} \leq |\omega_{11}| < 2^{\ell}) \\
&= \lim_{k\to\infty} \sum_{\ell=k}^{\infty} P(2^{\ell-1} \leq |\omega_{11}| < 2^{\ell}) \sum_{m=k}^{\ell} 2^{2m} \\
&\leq \lim_{k\to\infty} 2 \sum_{\ell=k}^{\infty} 2^{2\ell} P(2^{\ell-1} \leq |\omega_{11}| < 2^{\ell}) \leq \lim_{k\to\infty} 2\, E\, \omega_{11}^2 I_{[|\omega_{11}| \geq 2^{k-1}]} \\
&= 0 \qquad (2.16)
\end{aligned}
$$

which and (2.14) prove

$$J_3(n) \to 0. \quad \text{a.s.} \qquad (2.17)$$

(2.17) and (2.1) (2.6) (2.7) (2.11) complete the proof of Lemma 2.1.

Lemma 2.1 implies that for almost all ω, when n is large enough, we have

$$\sum_{k=0}^{\infty} \left|\left|\left(\frac{Wn}{n}\right)^k\right|\right| \leq \left|\left|\left(\frac{Wn}{n}\right)\right|\right| \sum_{k=0}^{\infty} \left|\left|\left(\frac{Wn}{n}\right)^2\right|\right|^k + \sum_{k=0}^{\infty} \left|\left|\left(\frac{Wn}{n}\right)^2\right|\right|^k < \infty.$$

Hence we obtain

Lemma 2.2. If $E\, \omega_{11} = 0$, $E\, \omega_{11}^2 < \infty$, then, almost surely, $(I - \frac{Wn}{n})^{-1}$ exists and equals $\sum_{k=0}^{\infty} (\frac{Wn}{n})^k$, when n is sufficiently large.

3. THE PROOF OF MAIN RESULTS

3.1. The proof of Theorem 1.

Firstly, we shall prove

$$\overline{\lim_{n\to\infty}} \left\|\frac{1}{n} W_n V_n\right\| \begin{cases} = 0 \quad \text{a.s. if } \lim\limits_{n\to\infty} \alpha_n = 0 \\ \leq M E^{\frac{1}{2}} \omega_{11}^2, \quad \text{a.s. if } \sup\limits_n |\alpha_n| \leq M \end{cases} \tag{3.1}$$

We have

$$\begin{aligned} \left\|\frac{1}{n} W_n V_n\right\|^2 &= \frac{1}{n^2} \sum_{k=1}^n \left(\sum_{i=1}^n \alpha_i \omega_{ki}\right)^2 \\ &= \frac{1}{n^2} \sum_{k=1}^n \sum_{i=1}^n \alpha_i^2 \omega_{ki}^2 + \frac{2}{n^2} \sum_{k=1}^n \sum_{i>j} \alpha_i \alpha_j \omega_{ki} \omega_{kj} \\ &= I_1(n) + I_2(n). \end{aligned} \tag{3.2}$$

By Kolmogorov's strong law of large numbers, we have for any m

$$I_1(n) \leq \frac{1}{n^2} \{M^2(1) \sum_{k=1}^n \sum_{i=1}^m \omega_{ki}^2 + M(m) \sum_{k=1} \sum_{i=m+1} \omega_{ki}^2 \}$$

$$\longrightarrow M^2(m)\, E\, \omega_{11}^2. \quad \text{a.s.}$$

where $M(m) = \sup\limits_{n\geq m} |\alpha_n| \leq M$, $m = 1,2,\ldots$. From this we can easily see that

$$\overline{\lim_{n\to\infty}}\, I_1(n) \begin{cases} = 0 & \text{if } \lim\limits_n \alpha_n = 0, \\ \leq M\, E\, \omega_{11}^2, & \text{a.s. if } \sup\limits_n |\alpha_n| \leq M \end{cases} \tag{3.3}$$

Let

$$Z_n = \sum_{k=1}^n \sum_{n\geq i>j\geq 1} \alpha_i \alpha_j \omega_{ki} \omega_{kj}.$$

Noting that $\{Z_n, n = 1,2,\ldots\}$ forms a martingale sequence, by a well-known martingale inequality, we get for any $\varepsilon > 0$

$$P\left(\max_{2^{m-1}\le n<2^m} \frac{2}{n^2}|Z_n| \ge \varepsilon\right) \le P\left(\max_{1\le n\le 2^m}|Z_n| \ge \frac{\varepsilon}{8}\, 2^{2m}\right)$$

$$\le \frac{8^2}{\varepsilon^2}\, 2^{-4m}\, E(Z_{2^m})^2$$

$$= 8^2\, \varepsilon^{-2}\, 2^{-4m} \sum_{k=1}^{2^m} \sum_{2^m\ge i>j\ge 1} \alpha_i^2\alpha_j^2\, (E\,\omega_{11}^2)^2$$

$$\le c\, 2^{-m}.$$

from which and Borel-Cantelli lemma, it follows that

$$I_2(n) = \frac{2}{n^2} Z_n \longrightarrow 0. \quad \text{a.s.} \tag{3.4}$$

By (3.2) (3.3) (3.4) we get (3.1).

Next we shall prove that

$$\left|\left|\left(\frac{Wn}{n}\right)^2 V_n\right|\right| \longrightarrow 0. \quad \text{a.s.} \tag{3.5}$$

We have

$$\left|\left|\left(\frac{Wn}{n}\right)^2 V_n\right|\right|^2 = \frac{1}{n^4}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\omega_{ij}\omega_{jk}\alpha_k\right)^2$$

$$\le \frac{2}{n^4}\left\{\sum_{i=1}^{n}\omega_{ii}^2\left(\sum_{k=1}^{n}\omega_{ik}\alpha_k\right)^2 + \sum_{i=1}^{n}\left(\sum_{\substack{j=1\\ j\ne i}}^{n}\sum_{k=1}^{n}\omega_{ij}\omega_{jk}\alpha_k\right)^2\right\} \tag{3.6}$$

We shall first prove that

$$\max_{1\le i\le n} n^{-3}\left(\sum_{k=1}^{n}\omega_{ik}\alpha_k\right)^2 \longrightarrow 0. \quad \text{a.s.} \tag{3.7}$$

If we have done so, then

$$\frac{2}{n^4}\sum_{i=1}^{n}\omega_{ii}^2\left(\sum_{k=1}^{n}\omega_{ik}\alpha_k\right)^2 \le 2\left(\frac{1}{n}\sum_{i=1}^{n}\omega_{ii}^2\right)\left(\max_{1\le i\le n} n^{-3}\left(\sum_{k=1}^{n}\omega_{ik}\alpha_k\right)^2\right)$$

$$\to 2\, E\,\omega_{11}^2 \cdot 0 = 0. \quad \text{a.s.} \tag{3.8}$$

Noting that $\{\sum_{k=1}^{n}\omega_{ik}\alpha_k,\ n = 1,2,\dots\}$ forms a martingale sequence, we have for any $\varepsilon > 0$

$$P\left(\max_{2^{m-1}\le n<2^m}\ \max_{1\le i\le n} n^{-3}\left(\sum_{i=1}^{n}\omega_{ik}\alpha_k\right)^2 \ge \varepsilon\right)$$

$$\le 2^m\, P\left(\max_{2^{m-1}\le n<2^m}\left(\sum_{i=1}^{n}\omega_{ik}\alpha_k\right)^2 \ge \varepsilon.\, 8^{-1}\, 2^{-3m}\right).$$

$$\le c\, 2^{-2m}\, E\left(\sum_{i=1}^{2^m}\omega_{ik}\alpha_k\right)^2 \le c\, M^2(1)\, 2^{-m} = c\, 2^{-m}. \tag{3.9}$$

which and Borel-Cantelli lemma imply (3.7). Hence, (3.8) holds.

On the other hand, we have

$$n^{-4}\sum_{i=1}^{n}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{k=1}^{n}\omega_{ij}\omega_{jk}\alpha_k\Big)^2$$

$$=n^{-4}\sum_{i=1}^{n}\Big\{\sum_{\substack{j=1\\ j\neq i}}^{n}\omega_{ij}^2\Big(\sum_{k=1}^{n}\omega_{jk}\alpha_k\Big)^2+2\sum_{\substack{j_1>j_2\\ j_1\neq i,j_2\neq i}}\sum\omega_{ij_1}\omega_{ij_2}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\omega_{j_1k_1}\omega_{j_2k_2}\alpha_{k_1}\alpha_{k_2}\Big\}$$

$$\triangleq n^{-4}\{R_1(n)+R_2(n)\}. \qquad (3.10)$$

Since

$$E(n^{-4}R_2(n))^2\leq c\,n^{-3},$$

we have

$$n^{-4}R_2(n)\longrightarrow 0.\quad \text{a.s.} \qquad (3.11)$$

Noticing that $\{\sum\sum_{n\geq k_1>k_2\geq 1}\omega_{1k_1}\omega_{1k_2}\alpha_{k_1}\alpha_{k_2},\ n=1,2,\ldots\}$ forms a martingale sequence, we obtain

$$P\Big(\max_{2^{m-1}\leq n<2^m}\max_{1\leq j\leq n}n^{-2}\Big|\sum\sum_{n\geq k_1>k_2\geq 1}\omega_{jk_1}\omega_{jk_2}\alpha_{k_1}\alpha_{k_2}\Big|\geq\varepsilon\Big)$$

$$\leq 2^m P\Big(\max_{n\leq 2^m}\Big|\sum\sum_{n\geq k_1>k_2\geq 1}\omega_{1k_1}\omega_{1k_2}\alpha_{k_1}\alpha_{k_2}\Big|\geq\frac{1}{4}\varepsilon\,2^{2m}\Big)$$

$$\leq c\,2^{-3m}E\Big(\sum\sum_{2^m\geq k_1>k_2\geq 1}\omega_{1k_1}\omega_{1k_2}\alpha_{k_1}\alpha_{k_2}\Big)^2$$

$$\leq c\,2^{-m},$$

which ensures that

$$\max_{1\leq j\leq n}n^{-2}\Big|\sum\sum_{n\geq k_1>k_2\geq 1}\omega_{jk_1}\omega_{jk_2}\alpha_{k_1}\alpha_{k_2}\Big|\to 0\quad \text{a.s.} \qquad (3.12)$$

Thus, to prove $n^{-4}R_1(n)\to 0$ a.s. we only need to prove

$$n^{-4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{n}\omega_{ij}^2\omega_{jk}^2\alpha_k^2\longrightarrow 0.\quad \text{a.s.} \qquad (3.13)$$

Since $|\alpha_k| \leq M$, it follows that we only need to prove

$$n^{-4} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sum_{k=1}^{n} \omega_{ij}^2 \omega_{jk}^2 \longrightarrow 0. \quad \text{a.s.} \tag{3.13}$$

This can be easily seen from the facts that $\{ \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1}^{n} \omega_{ij}^2 \omega_{jk}^2$, $n = 1,2,\ldots\}$ forms a semi-martingale sequence and that

$$P\left(\max_{2^{m-1} \leq n < 2^m} n^{-4} \left| \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1}^{n} \omega_{ij}^2 \omega_{jk}^2 \right| \geq \varepsilon \right)$$

$$\leq P\left(\max_{n \leq 2^m} \left| \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1}^{n} \omega_{ij}^2 \omega_{jk}^2 \right| \geq 2^{-4} \varepsilon\, 2^{4m} \right)$$

$$\leq c\, 2^{-4m} E \left| \sum_{i=1}^{2^m} \sum_{j=1,j\neq i}^{2^m} \sum_{k=1}^{2^m} \omega_{ij}^2 \omega_{jk}^2 \right|$$

$$\leq c\, 2^{-m}.$$

From (3.6), (3.7), (3.10), (3.11) and (3.13), we obtain (3.5).

By Lemma 2.2, for almost all ω, it holds that

$$X_n - V_n = \left(\frac{Wn}{n}\right) \left(I - \frac{1}{n} Wn\right)^{-1} V_n = \sum_{k=1}^{\infty} \left(\frac{Wn}{n}\right)^k V_n$$

$$= \left(\frac{Wn}{n}\right) V_n + \sum_{k=2}^{\infty} \left(\frac{Wn}{n}\right)^k V_n. \tag{3.16}$$

If $\lim_n \alpha_n = 0$, by (3.1), (3.5) and (3.16), we obtain

$$||X_n - V_n|| \leq \left(\left|\left| \left(\frac{Wn}{n}\right) V_n \right|\right| + \left|\left| \left(\frac{Wn}{n}\right)^2 V_n \right|\right| \right) \sum_{k=0}^{\infty} \left|\left| \left(\frac{Wn}{n}\right)^2 \right|\right|^k \to 0. \quad \text{a.s.} \tag{3.17}$$

which is equivalent to the second assertion of Theorem 1. Since

$$\left|\left| \sum_{k=2}^{\infty} \left(\frac{Wn}{n}\right)^k V_n \right|\right| \leq \left|\left| \left(\frac{Wn}{n}\right)^2 V_n \right|\right| \sum_{k=0}^{\infty} \left|\left| \left(\frac{Wn}{n}\right)^2 \right|\right|^k + \left|\left| \frac{Wn}{n} V_n \right|\right| \sum_{k\neq 1}^{\infty} \left|\left| \left(\frac{Wn}{n}\right)^2 \right|\right|^k$$

$$\longrightarrow 0. \quad \text{a.s.} \tag{3.18}$$

to prove the first assertion of Theorem 1, we only need to prove

$$\max_{1 \leq i \leq n} \left| \left(\frac{1}{n} W_n V_n\right)_i \right| = \max_{1 \leq i \leq n} \left| \frac{1}{n} \sum_{j=1}^{n} \omega_{ij}\, \alpha_j \right| \longrightarrow 0. \quad \text{a.s.} \tag{3.19}$$

or equivalently,

$$\max_{1 \leq i \leq n} \left| \frac{1}{n^2} \left(\sum_{i=1}^{n} \omega_{ij}\, \alpha_j \right)^2 \right| \longrightarrow 0. \quad \text{a.s.} \tag{3.20}$$

In view of (3.12), we only need to prove

$$\max_{1\le i\le n} \left| \frac{1}{n^2} \sum_{j=1}^{n} \omega_{ij}^2 \alpha_j^2 \right| \le M^2(1) \max_{1\le i\le n} n^{-2} \sum_{j=1}^{n} \omega_{ij}^2 \to 0. \text{ a.s.} \tag{3.21}$$

Set

$$Z_{jn} = \omega_{ij}^2 I_{[|\omega_{ij}| \le n]} - E\,\omega_{11}^2 I_{[|\omega_{11}| \le n]}$$

When m is so large that $E\,\omega_{11}^2 / 2^m < \frac{\varepsilon}{4}$, we have

$$P\Big(\max_{2^{m-1}\le n<2^m} \max_{1\le i\le n} n^{-2} \sum_{j=1}^{n} \omega_{ij}^2 \ge \varepsilon\Big)$$

$$\le 2^m P\Big(\max_{2^{m-1}\le n<2^m} \sum_{j=1}^{n} (\omega_{ij}^2 - E\,\omega_{11}^2 I_{[|\omega_{11}|<n]}) \ge \frac{\varepsilon}{8} 2^{2^m}\Big)$$

$$\le 2^m \Big[P\Big(\max_{2^{m-1}\le n<2^m} \sum_{j=1}^{n} Z_{jn} \ge \frac{\varepsilon}{8} 2^{2^m}\Big) + P\Big(\bigcup_{2^{m-1}\le n<2^m} \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (|\omega_{ij}| > n)\Big)\Big]$$

$$\le 2^m \Big[2^{-3m} E\, Z_{1\,2^m}^2 + P\Big(\bigcup_{i=1}^{2^m} \bigcup_{j=1}^{2^m} (|\omega_{ij}| > 2^{m-1})\Big)\Big]$$

$$\le 2^{-2m} E\,\omega_{11}^4 I_{[|\omega_{11}| \le 2^m]} + 2^{2m} P(|\omega_{11}| > 2^{m-1}). \tag{3.22}$$

where ε is an arbitrarily preassigned positive number.

From (2.16) we know

$$\sum_{m=1}^{\infty} 2^{2m} P(|\omega_{11}| > 2^{m-1}) < \infty. \tag{3.23}$$

On the other hand we have

$$\sum_{m=1}^{\infty} 2^{-2m} E\,\omega_{11}^4 I_{[|\omega_{11}| \le 2^m]} = \sum_{m=1}^{\infty} 2^{-2m} \Big[\sum_{k=1}^{m} E\,\omega_{11}^4 I_{[2^{k-1} < |\omega_{11}| \le 2^k]} + 1\Big]$$

$$\le \sum_{k=1}^{\infty} E\,\omega_{11}^4 I_{[2^{k-1} < |\omega_{11}| \le 2^k]} \sum_{m=k}^{\infty} 2^{-2m} + 1$$

$$\le 2 \sum_{k=1}^{\infty} 2^{-2k} E\,\omega_{11}^4 I_{[2^{k-1} < |\omega_{11}| \le 2^k]} + 1$$

$$\le 2\, E\,\omega_{11}^2 + 1 < \infty. \tag{3.24}$$

From (3.22), (3.23), (3.24) and Borel-Cantelli lemma, it follows that

$$\max_{1\le i\le n} n^{-2} \sum_{j=1}^{n} \omega_{ij}^2 \longrightarrow 0. \text{ a.s.} \tag{3.25}$$

The proof of Theorem 1 is completed.

3.2. The proof of Theorem 2.

Let M_n be the $n \times n$ matrix with all its entries being $m = E\,\omega_{11}$, and let $\hat{W}_n = W_n - M_n$. Write $\gamma_n = X_n - V_n\,/(1-m) = (X_{n,1} - \frac{\alpha}{1-m}, \ldots, X_{nn} - \frac{\alpha}{1-m})$. Then (1.1) can be rewritten as

$$\gamma_n = \frac{Wn}{n}\gamma_n + \frac{\alpha}{n(1-m)}\hat{W}_n 1$$

where $1 = (1,1,\ldots,1)^T$ being an $n \times 1$ vector.

Let $||A||_0$ denote the operator norm of the matrix A. Since $W_n = \hat{W}_n + M_n$, we have

$$||(\frac{Wn}{n})^2||_0 \le ||(\frac{\hat{W}n}{n})^2||_0 + ||\frac{\hat{W}nMn}{n^2}||_0 + ||\frac{Mn\hat{W}n}{n^2}||_0 + ||\frac{Mn^2}{n^2}||_0. \tag{3.26}$$

Applying lemma 2.1, we have

$$||(\frac{\hat{W}n}{n})^2||_0 \le ||(\frac{\hat{W}n}{n})^2|| \longrightarrow 0. \quad \text{a.s.,} \tag{3.27}$$

where $||A||$ denotes the Euclidean norm of the matrix A. We can easily compute that

$$||\frac{Mn^2}{n^2}||_0 = |m|\ ||\frac{Mn}{n}||_0 = \frac{m^2}{\sqrt{n}} \longrightarrow 0. \tag{3.28}$$

Furthermore, we have

$$||(\frac{\hat{W}n}{n})(\frac{Mn}{n})||_0^2 \le ||(\frac{\hat{W}n}{n})(\frac{Mn}{n})||^2$$
$$= \frac{m^2}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sum_{k=1}^{n}\hat{\omega}_{ik})^2 = \frac{m^2}{n^3}\sum_{i=1}^{n}(\sum_{k=1}^{n}\hat{\omega}_{ik})^2, \tag{3.29}$$

where $\hat{\omega}_{ik} = \omega_{ik} - m$ being a random variable with zero mean. Applying (3.12) with $\alpha_1 = \alpha_2 = \ldots = 1$, we have

$$\frac{2m^2}{n^3}\left|\sum_{i=1}^{n}\sum_{k_1>k_2}\hat{\omega}_{ik_1}\hat{\omega}_{ik_2}\right|$$
$$\le 2m^2 \max_{1\le i\le n} n^{-2}\left|\sum_{n\ge k_1>k_2\ge 1}\hat{\omega}_{ik_1}\hat{\omega}_{ik_2}\right| \to 0. \quad \text{a.s.} \tag{3.30}$$

By Kolmogorov's strong law of large numbers, we have

$$\frac{m^2}{n^3}\sum_{i=1}^{n}\sum_{k=1}^{n}\hat{\omega}_{ik}^{2}\longrightarrow 0\cdot \mathrm{Var}(\hat{\omega}_{11}) = 0. \quad \text{a.s.} \tag{3.31}$$

From (3.29), (3.30) and (3.31) we get

$$\left\|\frac{\hat{W}nMn}{n^2}\right\|_0 \longrightarrow 0. \quad \text{a.s.} \tag{3.32}$$

Similarly, we can prove that

$$\left\|\frac{Mn\hat{W}n}{n}\right\|_0 \longrightarrow 0. \quad \text{a.s.} \tag{3.33}$$

From (3.26), (3.27), (3.28), (3.32) and (3.33), we conclude that

$$\left\|\left(\frac{Wn}{n}\right)^2\right\|_0 \longrightarrow 0. \quad \text{a.s.} \tag{3.34}$$

Like proving Lemma 2.2, we see that for almost all ω, $(I-\frac{Wn}{n})^{-1}$ is nonsingular and

$$(I-\frac{Wn}{n})^{-1} = \sum_{k=0}^{\infty}\left(\frac{Wn}{n}\right)^k$$

when n is sufficiently large. Thus, for almost all ω, when n is large enough,

$$\begin{aligned}\gamma_n &= (1-\frac{Wn}{n})^{-1}\left(\frac{\alpha}{1-m}\,\frac{\hat{W}n}{n}\,\mathit{1}\right)\\ &= \frac{\alpha}{1-m}\sum_{k=0}^{\infty}\left(\frac{Wn}{n}\right)^k\left(\frac{\hat{W}n}{n}\right)\mathit{1}\,.\end{aligned} \tag{3.35}$$

Since

$$\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\omega}_{ij}\longrightarrow 0. \quad \text{a.s. (from Marcinkiewicz theorem)}$$

we have

$$\left\|\frac{Mn}{n}\,\frac{\hat{W}n}{n}\,\mathit{1}\right\|^2 = m^2n^{-3}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\omega}_{ij}\right)^2\longrightarrow 0. \quad \text{a.s.} \tag{3.36}$$

from which and (3.5) we conclude that

$$\left\|\frac{Wn}{n}\,\frac{\hat{W}n}{n}\,\mathit{1}\right\| \le \left\|\left(\frac{\hat{W}n}{n}\right)^2\mathit{1}\right\| + \left\|\frac{Mn}{n}\,\frac{\hat{W}n}{n}\,\mathit{1}\right\| \to 0. \quad \text{a.s.} \tag{3.37}$$

Applying (3.1) with M = 1, we have

$$\overline{\lim_{n\to\infty}}\left\|\frac{\hat{W}n}{n}\,\mathit{1}\right\| \le E^{\frac{1}{2}}\,\hat{\omega}_{11}, \text{ a.s.} \tag{3.38}$$

Consequently, by (3.37) and (3.38),

$$\left|\left|\sum_{k=1}^{\infty}\left(\frac{Wn}{n}\right)^k\left(\frac{\hat{Wn}}{n}\right) 1\right|\right|$$

$$\leq \left|\left|\frac{\hat{Wn}}{n} 1\right|\right| \sum_{k=1}^{\infty} \left|\left|\left(\frac{Wn}{n}\right)^2\right|\right|_0^k + \left|\left|\frac{Wn}{n}\ \frac{\hat{Wn}}{n} 1\right|\right| \sum_{k=0}^{\infty} \left|\left|\left(\frac{Wn}{n}\right)^2\right|\right|_0^k$$

$$\longrightarrow 0. \quad \text{a.s.} \tag{3.39}$$

Using the same method as used in the proof of Theorem 1, we can prove

$$\max_{1\leq j\leq n} \left|\frac{1}{n}\sum_{i=1}^{n}\hat{\omega}_{ij}\right| \longrightarrow 0. \quad \text{a.s.} \tag{3.40}$$

which and (3.35), (3.39) imply Theorem 2, and the proof is finished.

3.3. The proof of Theorem 3.

Without loss of generality, we can assume that $m_i = i$, $i=1,2,\ldots,k$. In the proof of Theorem 1, we have shown that, for almost all ω, when n is large enough,

$$X_n - V_n = \sum_{k=1}^{\infty}\left(\frac{Wn}{n}\right)^k V_n. \tag{3.41}$$

Note that

$$\left\{ \frac{n}{\sigma||V_n||} \left(\frac{Wn}{n}\right) V_n\Big|_j,\ j=1,2,\ldots,k \right\}$$

$$= \left\{ \frac{1}{\sigma||V_n||} \sum_{\ell=1}^{n} \alpha_\ell \omega_{j\ell},\ j=1,2,\ldots,k \right\}$$

is a set of independent and identically distributed random variables, each of which is a normalized sum of independent random variables, where $(X_1,X_2,\ldots,X_n)^T{}_j = X_j$. Since $\{\alpha_i,\ i=1,2,\ldots\}$ is bounded, and $\sum_{i=1}^{\infty}\alpha_i^2 = \infty$. According to the Lindeberg-Feller theorem, we have

$$\left\{ \frac{n}{\sigma||V_n||} \left(\frac{Wn}{n}\right) V_n\Big|_j,\ j=1,2,\ldots,k\right\} \xrightarrow{d} N(0,I_k). \tag{3.42}$$

We have

$$\left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^2 V_n \right|\right|^2$$

$$= \frac{1}{n^2||V_n||^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \sum_{\ell=1}^{n} \omega_{ij}\omega_{j\ell}\alpha_\ell \right)^2$$

$$\leq \frac{2}{n^2||V_n||^2} \left\{ \sum_{i=1}^{n} \omega_{ii}^2 \left(\sum_{\ell=1}^{n} \omega_{i\ell}\alpha_\ell \right)^2 + \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\ell=1}^{n} \omega_{ij}\omega_{j\ell}\alpha_\ell \right)^2 \right\}$$

$$\leq \frac{4}{n^2||V_n||^2} \left\{ \sum_{i=1}^{n} \omega_{ii}^4\alpha_i^2 + \sum_{i=1}^{n} \omega_{ii}^2 \left(\sum_{\substack{\ell=1 \\ \ell\neq i}}^{n} {}_{i\ell}\alpha_\ell \right)^2 + \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\ell=1}^{n} \omega_{ij}\omega_{j\ell}\alpha_\ell \right) \right\}$$

$$= J_1(n) + J_2(n) + J_3(n). \tag{3.43}$$

It is obvious that

$$J_1(n) \leq 4\, n^{-2} \sum_{i=1}^{n} \omega_{ii}^4 \longrightarrow 0. \quad \text{a.s. (see, Marcinkiewicz theorem)} \tag{3.44}$$

and

$$E\, J_2(n) \leq 4(E\, \omega_{11}^2)^2 / n \longrightarrow 0, \tag{3.45}$$

and

$$E\, J_3(n) \leq 4(E\, \omega_{11}^2)^2. \tag{3.46}$$

Hence

$$\left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^2 V_n \right|\right| = 0_p(1), \; n \longrightarrow \infty \tag{3.47}$$

where "$||X_n|| = 0_p(1),\ n \to \infty$" means that the sequence $\{X_n,\ n = 1,2,\ldots\}$ are uniformly bounded in probability in the sense of Euclidean norm.

Since $\left|\left|\frac{W_n}{n}\right|\right|^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij}^2 \to E\, \omega_{11}^2$, a.s.

we have

$$\left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^3 V_n \right|\right| \leq \left|\left|\frac{W_n}{n}\right|\right| \left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^2 V_n \right|\right| = 0_p(1),\ n \to \infty,$$

which and (3.47) and Lemma 2.1 imply

$$\left|\left| \frac{n}{||V_n||} \sum_{\ell=4}^{\infty} \left(\frac{W_n}{n}\right)^\ell V_n \right|\right| \leq \left\{ \left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^2 V_n \right|\right| + \left|\left| \frac{n}{||V_n||} \left(\frac{W_n}{n}\right)^3 V_n \right|\right| \right\}$$

$$\cdot \left|\left| \left(\frac{W_n}{n}\right)^2 \right|\right| \sum_{\ell=o}^{\infty} \left|\left| \left(\frac{W_n}{n}\right)^2 \right|\right|^\ell \longrightarrow 0. \quad \text{in p.} \tag{3.48}$$

By (3.41), (3.42), (3.48), to prove Theorem 3, we only need to prove

$$\frac{n}{||V_n||} \left(\frac{Wn}{n}\right)^{\ell} V_n\Big|_j \longrightarrow 0. \quad \text{in p.} \quad \ell = 2,3, \; j=1,2,\ldots,k \tag{3.49}$$

Since for each ℓ, $\frac{n}{||V_n||} \left(\frac{Wn}{n}\right)^{\ell} V_n\Big|_j$, $j = 1,2,\ldots,k$, are identically distributed, we only need to prove (3.49) for $j = 1$.

For $\ell = 2$, we have

$$\frac{1}{n||V_n||} W_n^2 V_n \Big|_1 = \frac{1}{n||V_n||} \sum_{i=1}^{n} \sum_{\ell=1}^{n} \omega_{1i}\, \omega_{i\ell}\, \alpha_\ell$$

$$= \frac{1}{n||V_n||} \{\omega_{11}^2 \alpha_1 + \omega_{11} \sum_{\ell=2}^{n} \omega_{i\ell}\alpha_\ell + \sum_{i=2}^{n} \sum_{\ell=1}^{n} \omega_{1i}\, \omega_{i\ell}\, \alpha_\ell\}$$

$$= J_1(n) + J_2(n) + J_3(n) \tag{3.50}$$

It is obvious that

$$|J_1(n)| \leq \frac{1}{n} \omega_{11}^2 \longrightarrow 0. \quad \text{a.s.} \tag{3.51}$$

and

$$E(J_2(n)) = \frac{1}{n^2||V_n||^2} \sum_{\ell=2}^{n} \alpha_\ell^2 (E\, \omega_{11}^2)^2 \leq \frac{(E\, \omega_{11}^2)^2}{n^2} \tag{3.52}$$

$$E(J_3(n)) = \frac{1}{n^2||V_n||^2} \sum_{i=2}^{n} \sum_{\ell=1}^{n} \alpha_\ell^2 (E\, \omega_{11}^2)^2$$

$$\leq \frac{(E\, \omega_{11}^2)^2}{n} \tag{3.53}$$

From (3.52) and (3.53), we obtain

$$J_2(n) \longrightarrow 0. \quad \text{a.s.}$$

$$J_3(n) \longrightarrow 0, \quad \text{in p.}$$

which and (3.51) imply (3.49) for $\ell = 2$.

For $\ell = 3$ we have

$$\frac{n}{||V_n||} \left(\frac{Wn}{n}\right)^3 V_n\Big|_1 = \frac{1}{n^2||V_n||} \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \omega_{1i}\, \omega_{i\ell}\, \omega_{\ell m}\, \alpha_m$$

$$= \frac{1}{n^2||V_n||} \{\omega_{11}^3 \alpha_1 + \sum_{i=2}^{n} \omega_{1i}\, \omega_{i1}^2\, \alpha_i + \sum\nolimits_{(2)} \omega_{1i}\, \omega_{i\ell}\, \omega_{\ell m}\, \alpha_m$$

$$+ \sum\nolimits_{(3)} \omega_{1i}\, \omega_{i\ell}\, \omega_{\ell m}\, \alpha_m \} \tag{3.54}$$

where $\sum_{(2)}$ runs over the set (i,ℓ,m); $2 \le i \le n$, $1 \le \ell$, $m \le n$, two of (i,ℓ,m) are equal to each other, but the other one is not equal to them.}, and $\sum_{(3)}$ runs over the set $\{(i,\ell,m)$, $2 \le i \le n$, $1 \le \ell$, $m \le n$, any two of (i,ℓ,m) are not equal to each other.}

It is obvious that

$$\frac{1}{n^2||V_n||} \omega_{11}^3 \alpha_1 \longrightarrow 0,$$

$$E\left|\frac{1}{n^2||V_n||} \sum_{i=2} \omega_{1i}\omega_{ii}^2 \alpha_i\right| \le \frac{M}{n||V_n||} E|\omega_{11}| E\, \omega_{11}^2 \longrightarrow 0,$$

$$E\left|\frac{1}{n^2||V_n||} \sum_{(2)} \omega_{1i}\, \omega_{i\ell}\, \omega_{\ell m}\, \alpha_m\right| \le \frac{3\,M\,E|\omega_{11}|\, E\omega_{11}^2}{||V_n||} \longrightarrow 0,$$

and

$$E\left(\frac{1}{n^2||V_n||} \sum_{(3)} \omega_{1i}\, \omega_{i\ell}\, \omega_{\ell m}\, \alpha_m\right)^2 \le (E\,\omega_{11}^2)^3 / n^2 \longrightarrow 0,$$

where M is the super bound of the sequence $\{\alpha_1, \alpha_2, \ldots\}$, which and (3.54) imply (3.49) for $\ell = 3$, and the proof of Theorem 3 is proved.

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